IJFBS
Finance & Banking Studies

Finance & Banking Studies

IJFBS, VOL 10 NO 3 ISSN: 2147-4486

Contents available at www.ssbfn.net/ojs

<https://doi.org/10.20525/ijfbs.v10i3.1278>

The application of different term-structure models to estimate South African real spot rate curve

Mmakganya Mashoene

Corresponding Author: Department of Finance, University of KwaZulu-Natal, Durban, South Africa. +27 78 558 5502, ORCID ID: 0000-0002-4789-7113

Mishelle Doorasamy

Department of Accounting, University of KwaZulu-Natal, Durban, South Africa. +27 31 260 2155, ORCID ID: 0000-0001-9320-3461

Rajendra Rajaram

Department of Management Accounting, University of KwaZulu-Natal, Durban, South Africa. +27 33 260 6267, ORCID ID: 0000-0003-1734-2448

Abstract

The purpose of this study is to investigate the suitable arbitrage-free term-structure model that might be able to fit the South African inflation-indexed spot-rate curve. The instrument has relatively less tradability in the market, which then translates into a lack of adequate data for bond valuation/pricing. Pricing deviations might give inflated/deflated projections on the value of government debt; consequently, higher estimated interest cost to be paid. A proper valuation of these instruments is mandatory as they form part of government funding/borrowing and the country's budgeting processes in the medium term. The performance of newly developed non-linear multifactor models that follows the Nelson-Siegel (1987) framework was compared to the arbitrage-free Vasicek (1977) model and linear parametric models to assess any significant deviations in forecasting the real spot-rate curve over a short period. Models with constant parameters (i.e. linear parametric, cubic splines, Nelson-Siegel (1987) and Svensson (1994)) gave a perfect fit, they proved to marginally lose fitting capabilities during periods of higher volatility. Therefore, it could be concluded that the application of either Nelson-Siegel (1987) model or Svensson (1994) model on forecasting South African real spot-rate curve gave a perfect fit. However, for a solid conclusion to be derived, it is imperative to explore the performance of these models over a period of stressed market and economic conditions.

Keywords: South African inflation-indexed bonds; Parametric yield curve models; Arbitrage-free generalised Nelson Siegel model; Illiquid bond markets; Rotated Dynamic Nelson-Siegel model

JEL Classifications: G12; G32; H63

Introduction

Harman and Mageza (2009, pp. 3) indicated that South African government introduced the inflation-linked debt market as recently as March 2000. The introduction of the market was to entice investors into investing in the country by introducing a product that hedges their investments against inflation in line with monetary policy strategy on inflation targeting. However, the introduction of the inflation-linked debt became more expensive for the South African government when a 13-year R189 bond was launched at a real coupon rate of 6.25 per cent in the year 2000 (Bloomberg, 2018). The consumer price inflation rate is not included in the yield-to-maturity rate charged for this product, thus reflecting what is perceived by the market in the real economy [i.e. Garcia and van Rixtel (2007, pp.34) indicated that real spot-rates are perceived as an indicator of the real economic growth in the future]. The market continued to grow with the introduction of a 22-year R197 bond at 5.5 per cent coupon rate in the year 2001, a 6-year R198 bond at 3.8 per cent coupon rate in the year 2002 and a 33-year R202 bond at 3.45 per cent coupon rate in 2003, (Bloomberg, 2018).

The cost of borrowing continued to reprice in line with the popularity of these instruments in the market; thus, reflecting a real macroeconomic environment in South Africa. The popularity has developed to a point where the inflation-linked bond market contributes over 22 per cent of total domestic government debt as at the end of the 2016/17 fiscal year (National Treasury, 2017). As such, at the beginning of March 2017, a range of between 2.03 and 2.13 per cent is observed on the real coupon-bearing yields across the entire curve. Stronger real coupon-bearing yields, when compared to the period when the instrument was initially introduced by South African government in early 2000, implies that the inflation-linked debt market has gained popularity and its demand base has also developed to a point where this market contributes over 22 per cent of total domestic government debt (National Treasury, 2017). According to the National Treasury (2017), official pension funds hold the highest 47 per cent of ILBs, followed by monetary authorities and private self-administered funds at 21 per cent and 15 per cent, respectively. Even though the inflation-linked bond market in South Africa continued to grow to a reasonable amount of R443 billion (revalued) issued as at March 2017 (National Treasury, 2017), they have not yet gained much momentum in terms of tradability as investors buy to hold the bonds. Given the illiquid nature of inflation-linked bonds, pricing the real spot-rate curve remains critical; as such, Johannesburg Stock Exchange (2012) established a model to estimate the real spot-rate curve given the coupon-bearing inflation-linked bond prices.

The Johannesburg Stock Exchange (2012) methodology is also applied by National Treasury when introducing a new bond in the market (National Treasury, 2012). Given the lengthy process embedded in this methodology, which involves deriving the real overnight rate from the nominal overnight rate (as a proxy for a non-existent short-rate), it might not be expected for National Treasury to price the real spot-rate curve; as such, there is much dependence on the output given by JSE for the pricing of the newly introduced bond instrument. Furthermore, for budgeting and risk management processes, the existence of a real spot-rate pricing model to price benchmark spot-rates is equally critical. The current econometrics methodology used by the National Treasury in this regard is based on the Taylor rule (Bold and Harris, 2018). It has the capability of giving only two points on the real spot-rate curve (i.e. the 3-month and 10-year benchmark points). Based on National Treasury (2018), it could be observed that the bulk of the issuance (at least 83 percent) was in the ultra-long end of the real curve (i.e. with at least ten years maturity). The allocation on this space of the curve poses gaps in the pricing of inflation-linked bonds used for borrowing given that either an interpolation process has to be done to get yields on longer maturities or yields have to be kept constant on this maturity spectrum (Naidoo et al., 2020).

A parametric mathematical term-structure model with fewer parameters to determine the entire real spot-rate curve will be explored in this study to examine its suitability in pricing inflation-linked bonds. The process seeks to minimise the model risk that might emanate from; firstly, the considerable work that goes into estimating the real spot-rate curve based on Johannesburg Stock Exchange (2012) methodology; and secondly, the risk of mispricing the ultra-long-end of the curve due to only a few points estimated on the entire real spot-rate curve. The paper is divided into four sections; firstly, the literature review which details recent work done on the South African real yield curve and further exploring the work done on negative yields and illiquid bond markets. The second section details different developments made on the Nelson-Siegel (1987) term-structure model which are popular in the market; and the methodology used to fit, estimate and forecast the South African real spot-rate curve given the selected term-structure models. The third section details the fitting output and the analysis of the output; and the economic significance of the output in modelling the South African spot-rate curve. The fourth section concludes the analysis by giving economic recommendations and suggestions for future studies.

Literature Review

The South African inflation-linked (real) bond market is very narrow and has a presence of a couple of gaps (i.e. fewer maturities, less liquidity, and the newness of the inflation-linked bond market). This might have contributed to a lack of many empirical studies in trying to construct the real spot-rate curve. The current South African real spot-rate curve on the JSE is estimated based on the Johannesburg Stock Exchange (2012) methodology. The methodology is based on the model implied spot-rate curve given the coupon-bearing inflation-indexed bond prices. The first step in estimating the real spot-rate curve is finding the real overnight rate. This is non-existent given that there is no overnight inflation-indexed securities trading on the JSE. This methodology thus models the real overnight rate given the nominal overnight rate $\dot{r}_t$:

$$\hat{r}_t = 365 \left[\frac{CPI_{t_0} (1 + \dot{r}_t / 365)}{CPI_{t_0+1}} - 1 \right]$$

Given the theoretical all-in price for the inflation-linked bond as:

$$\hat{B}(t, \tau) = \Gamma \left[e^{r_{settle} \tau_{settle}} \sum_{i=1}^n CF_i e^{-r_i \tau_i} \right]$$

where:

- Γ is the inflation indexation factor for a bond at time τ_{settle} . It is defined as the division of the reference CPI corresponding to τ_{settle} (Rafaelli, 2006, pp.8) by the bond specific base index.
- CF_i is the non-indexed cash flows occurring at term-to-maturity τ_i for $i \in \{1, 2, \dots, n\}$
- r_i is the real spot-rate applicable at term-to-maturity τ_i for $i \in \{1, 2, \dots, n\}$

Then the real spot-rate is the modelled as:

$$\hat{r}_{\tau_n} = \frac{1}{\tau_n} \left[\ln CF_n - \ln \left(\frac{B(t, \tau)}{\Gamma} \cdot e^{-r_{settle} \tau_{settle}} - \sum_{i=1}^{n-1} CF_i e^{-r_i \tau_i} \right) \right]$$

To estimate the spot-rate for maturity τ_i for $i \in \{1, 2, \dots, n\}$, the calibration exercise is performed starting with the shortest maturity until the entire real spot-rate curve is calibrated. Even though this methodology is consistent with the pricing of the inflation-indexed bond (Rafaelli, 2006, pp.8); it could be noted that since the quantity $\frac{CPI_{t_0}}{CPI_{t_0+1}}$ is seasonal, this could result with some form of instability on shorter maturities.

The most recent academic work on finding a mathematical methodology to estimate the South African real forward-rate curve was done by Reid (2009). The methodology was attempting to use forward interest rates on the South African financial market to isolate a measure of inflation expectations. Based on the *Fisher equation*, inflation expectation is modelled as:

$$\pi_{\tau}^e = \ddot{R}(t, \tau) - R(t, \tau)$$

where $\ddot{R}(t, \tau)$ is the nominal spot-rate, $R(t, \tau)$ is the real spot-rate, π_{τ}^e is future inflation expectation for term-to-maturity τ at time t .

Figure 1: Selected European countries yield curves (Carvalho and Garcia, 2019, pp. 18)

Figure 2: Yield curves as of March 17, 2017 (Lartey and Li, 2018, pp. 13)

To estimate future inflation expectations, nominal and real forward rates were modelled separately. In estimating forward rates (both nominal and real), the Svensson model was opted for. It was due to its advantage of imposing a second curvature in the long-end of the curve to address the issue of constant forward-rates on longer maturities which is embedded in the Nelson-Siegel's methodology; Svensson

(1994). Deviations between actual prices and model estimated prices were minimized using maximum likelihood methodology to estimate the optimal parameters β that will be able to fit the South African forward rate curve. It was indicated that the fitting process was fairly reasonable as predicted by the literature. Most of the time, the real forward rate curve gave a flatter projections on longer maturities with a variation of between 2 and 5 per cent over the entire estimation period. This is in-line with the results obtained from the work done by Carvalho and Garcia (2019) on the Eurozone by fitting the negative yields on the Svensson (1994) model (refer to *Figure 1*); and also the work by Larrey and Li (2018) on the Ghanaian bond market where the Svensson (1994) model was one of the best fit for the illiquid bond market (refer to *Figure 2*).

Given that the Johannesburg Stock Exchange (2012) methodology requires the use of the non-existent real short-rate, this poses a limitation in the estimation of the South African real spot-rate curve. The work done by Reid (2009) is a step in a right direction and this study will be focusing on exploring different term structure models to fit, estimate and forecast the South African real spot rate curve. The inflation-linked bonds are currently used for government funding and a reliable term-structure model is imperative to supplement the use of the Johannesburg Stock Exchange (2012) methodology for budgeting purposes.

Research and Methodology

The main purpose of this study is to find a suitable term-structure model to fit the South African real spot-rate curve. As such, a literature review on mathematical term-structure models that are commonly used in the markets is imperative to gain an understanding of the characteristics embedded in each model. For the analysis of the real spot-rate curve, continuously compounded real spot-rates are used. The real spot-rate curve is usually preferred for term-structure analysis because of its lesser maturity specific risk imposed by coupon rates. However, the real spot-rate curve is often not available in most countries (i.e. South African financial market where the bonds are currently not issued), or there are only fewer points available on the entire maturity profile. The following frameworks are of interest in term-structure modelling:

Vasicek (1977) one-factor model has a mean-reversion property (Hull, 2009). This implies that the short-rate is pulled back to the long-term average at a specified mean-reversion rate such that yields do not persist to be weak/strong over a longer period. The model allows for negative spot-rates; as such, it was selected due to often observed negative coupon-bearing real yields in the short-end of the South African inflation-linked bond market. Under the arbitrage-free or the risk-neutral world, the spot-rate is defined as:

$$\hat{R}(t, \tau) = - \left[\ln[\hat{P}(t, \tau)] / \tau \right]$$

where

$$\begin{aligned} \hat{P}(t, \tau) &= B_t(T) e^{-A_t(T)r_t} \\ r_t &= r_0 e^{-kt} + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{-k(t-u)} dW_u \\ A_t(T) &= \frac{1}{k} (1 - e^{-k\tau}) \\ B_t(T) &= e^{\left\{ \frac{(\sigma^2 - k^2\theta + \sigma k\lambda(t, r_t))}{k^2} [A_t(T) - \tau] - \frac{\sigma^2}{4k} A_t(T)^2 \right\}} \end{aligned}$$

The spot-rate curve derived under the arbitrage-free Vasicek one-factor model uses the short-rate to derive the entire spot-rate curve. This might not be able to perfectly derive the South African inflation-indexed spot-rate curve, mainly due to the illiquid nature of these instruments and the shape of the curve that might not be perfectly derived under one-factor term-structure models. This then raises the issue of constructing the real spot-rate curve from the coupon-bearing inflation-linked bond prices.

Functions where term-to-maturity is used as a parameter are one of the most popular models used to address the issue of a non-invertible cash flow matrix which might arise when a number of future cash flows is not the same as the number of bonds available. These functions are used to derive the discount curve that is a function of term-to-maturity (i.e. $T - t = \tau$) and other parameters (i.e. $a, b, \dots$). This then will define the discount curve function as $d(\tau) = d(\tau; a, b, \dots)$. The most popular and easy to use parameterised yield curve models are linear models which are defined as a linear combination of some basis elements.

The other popular method used in the market is the spline yield-curve model. The model also uses a linear function and it is considered easy to fit (Suli, 2014). Even though it uses a linear function, there is a relatively larger number of parameters to be estimated to construct the entire discount factor curve. This is because it uses knot points to join different polynomial functions on different knots across the entire curve. For a spline of polynomial order m with $n - 1$ knot points will need $m(n - 1)$ parameters to construct the discount factor curve. The ability to fit a curve of a very complex shape by using the knots to break the curve into differentiable polynomial functions might give the splines models a better advantage.

In trying to rectify draw-backs that are possible from arbitrage-free traditional models, non-linear parametric Nelson-Siegel (1987) model is proposed to model the forward rate using three latent factors by employing a relationship from expectation theory. The main advantage of this method is that it ensures a smooth and fairly flexible curve, Nelson and Siegel (1987, pp.474-481).

$$R(t, \tau) = \beta_0 + \beta_1 \left[\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right] + \beta_3 \left[\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right]$$

Nelson-Siegel-Svensson/ Svensson (1994) model was first developed to address the in-sample shortcomings (for the given level of λ) highlighted on the Nelson-Siegel (1987) model [i.e. the factor loading for the slope and the curvature factor decay rapidly to zero as a function of maturity]. Christensen et al (2009, pp.6) indicated that, the level factor is the only factor capable of fitting spot-rates in the maturities spectrum of ten years and above; thus, showing a lack of a perfect fit for long-term spot rate. Svensson (1994) model introduced an extension of Nelson-Siegel (1987) by incorporating the second curvature with the corresponding decay rate; as such, the forward rate was then presented as:

$$R(t, \tau) = \beta_0 + \beta_1 \left[\frac{1 - e^{-\lambda_1\tau}}{\lambda_1\tau} \right] + \beta_2 \left[\frac{1 - e^{-\lambda_1\tau}}{\lambda_1\tau} - e^{-\lambda_1\tau} \right] + \beta_3 \left[\frac{1 - e^{-\lambda_2\tau}}{\lambda_2\tau} - e^{-\lambda_2\tau} \right]$$

Increased parameters on the Svensson (1994) model have enabled the model to be able to capture relatively more complex shapes than the Nelson-Siegel (1987) model. However, as indicated by Carvalho and Garcia (2019), the main disadvantage of the Svensson (1994, pp.2) model is its ability to produce negative spot-rates and the constant parameters over time. This might not be able to capture the dynamics embedded in financial markets over time. As such, Diebold and Li (2006, pp.341) introduced a time-dependent Nelson-Siegel (1987) model as follows:

$$R(t, \tau) = \beta_{0,t} + \beta_{1,t} \left[\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right] + \beta_{3,t} \left[\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right]$$

Christensen et al. (2009, pp.7) suggested the introduction of a dynamic generalized Nelson-Siegel (DGNS) model by incorporating a second slope to produce a five factor loading structure. This is because the one-slope factor loading setting in the Svensson (1994) model might not be able produce an arbitrage-free property with a two-curvature factor loading structure.

$$R(t, \tau) = \beta_{0,t} + \beta_{1,t} \left[\frac{1 - e^{-\lambda_1\tau}}{\lambda_1\tau} \right] + \beta_{2,t} \left[\frac{1 - e^{-\lambda_2\tau}}{\lambda_2\tau} \right] + \beta_{3,t} \left[\frac{1 - e^{-\lambda_1\tau}}{\lambda_1\tau} - e^{-\lambda_1\tau} \right] + \beta_{4,t} \left[\frac{1 - e^{-\lambda_2\tau}}{\lambda_2\tau} - e^{-\lambda_2\tau} \right]$$

However, even-though the DGNS factors are chosen to have stochastic dynamics; it is impossible to prevent arbitrage opportunities using the bond prices implied by the resulting Nelson-Siegel (1987) yield curve. This weakness is therefore overcome by the introduction of the Arbitrage-Free Generalized Nelson-Siegel/ AFGNS (2009) model. Christensen et al. (2009, pp.1) solved this problem by "reconciling the Nelson-Siegel (1987) model with the absence of arbitrage by deriving an affine arbitrage-free model that maintains the Nelson-Siegel (1987) factor loading structure for the spot-rate curve".

$$R(t, \tau) = X_t^1 + X_t^2 \left[\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} \right] + X_t^3 \left[\frac{1 - e^{-\lambda_2 \tau}}{\lambda_2 \tau} \right] + X_t^4 \left[\frac{1 - e^{-\lambda_1 \tau}}{\lambda_1 \tau} - e^{-\lambda_1 \tau} \right] \\ + X_t^5 \left[\frac{1 - e^{-\lambda_2 \tau}}{\lambda_2 \tau} - e^{-\lambda_2 \tau} \right] - \frac{C_t(T)}{\tau}$$

where $-\frac{C_t(T)}{\tau} = -\frac{1}{2\tau} \int_t^T \sum_{j=1}^5 [\Sigma' B_u(T) B_u(T)' \Sigma]_{j,j} du$

Nyholm (2018, pp.114) then proposed a new development on the Dynamic Nelson-Siegel (2006) model where yield curve factors are rotated such that the short-rate will be part of the factors modelled. The Rotated Dynamic Nelson-Siegel/RDNS (2018) model was then developed where the exogenous variables that might affect the dynamics of the short-rate are integrated into the model. Given the Dynamic Nelson-Siegel (2006) model, the spot-rate curve is defined in a matrix form as

$$R(t, \tau) = \theta \cdot \beta + \varepsilon \text{ and } \beta_t = C + A x_{t-1} + v_t$$

where $\varepsilon \sim N(0, \sigma^2)$,

$$\theta = \begin{bmatrix} 1 & \frac{1-e^{-\lambda \tau_1}}{\lambda \tau_1} & \left[\frac{1-e^{-\lambda \tau_1}}{\lambda \tau_1} - c_t e^{-\lambda \tau_1} \right] \\ \vdots & \vdots & \vdots \\ 1 & \frac{1-e^{-\lambda \tau_n}}{\lambda \tau_n} & \left[\frac{1-e^{-\lambda \tau_n}}{\lambda \tau_n} - c_t e^{-\lambda \tau_n} \right] \end{bmatrix}, \beta = \begin{bmatrix} \beta_{0,t} \\ \beta_{1,t} \\ \beta_{2,t} \end{bmatrix}$$

Then the spot-rate curve under the Rotated Nelson-Siegel (2018) model is defined as:

$$R(t, \tau) = G \cdot \gamma + \varepsilon$$

where D is a rotation matrix, then

$$G = \theta \cdot D^{-1} \\ \gamma = D \cdot \beta \\ m = D \cdot C \\ F = D \cdot A \cdot D^{-1} \\ z_t = A \cdot v_t$$

Data analysis

The monthly analysis covered the months of April 2016 and May 2016 and could be deduced that over the two months, there is no significant difference in the mid-end of the curve. Furthermore, an analysis on the quarterly data covered the two quarters of 2016 (i.e. first quarter- May, June, July; and second quarter- August, September, October). It could then be indicated that on a quarterly basis, there is a significant difference in sample means. It was then decided that quarterly data covering the period from April 2010 to October 2018 will be used in this study. The development phase will cover the period from April 2010 to April 2018, the testing period will cover the period from May 2018 to October 2018. The quarterly data will be presented as Q1: February, March, April; Q2: May, June, July, Q3: August, September, October; and Q4: November December, January. This is a sample of 34 data points for development and 2 data points for testing. Development data is sufficient to make a significant conclusion based on the Central Limit Theorem (CLT) which indicates that as the sample size increases, the mean of the sample will be closer to the mean of the population (Gujarati, 2009). Furthermore, the sample size of at least 30 data points is deemed sufficient for CLT to hold. Three different types of data will be used for the analysis. These include inflation-linked bond information; all-in inflation-linked bond prices and bond liquidity data.

Modelling process

The modelling process will involve OLS (Ordinary Least-Squares) methodology to obtain optimal model parameters that should minimize the squared error between the model and actual bond prices. The OLS is a statistical methodology that analyses the relationship by minimizing the squared difference between the actual and observed variables, Gujarati (2009). To extract the continuously compounded spot-rate curve assuming

arbitrage-free environment, the inflation-indexed coupon-bearing bond all-in price and discount factors should satisfy

$$\hat{B}(t, \tau) = \sum_{t=0.5}^{\tau} [CF(t, \tau) \cdot d(\tau)]$$

where $d(\tau) = e^{-R(t, \tau) \cdot \tau}$ is a discount function using continuously compounded real spot-rates for inflation-indexed bonds and $CF(\tau, \tau) = 100 + Coupon_i$ is the cashflow at maturity which comprises of the par value of 100 and the coupon-payment of bond i . This implies that a set of inflation-indexed coupon-bearing bond prices in the absence of arbitrage opportunities can be set-up in the following matrix form

$$B = CF \cdot d + e_t$$

Given that the discount function is an exponential function, a log-transformation is applied to approximately conform to normality. Applying the OLS methodology by minimizing squared errors, the optimization function is defined as

$$\min_{\beta} e_t^T e_t = (B - CF \cdot d)^T (B - CF \cdot d)$$

where β is a set of estimate parameters to be optimized.

Findings

Estimation method selection process entails some detailed tests that each model must pass to be considered as a good model that will be able to fit the data in question perfectly. The selection method of the best mathematical term-structure model was based on the following aspects of a spot-rate curve, which was also applied in Kikuchi and Shintani (2012, pp.30).

Negative spot-rates (refer to Table 1) imply that investors are not being compensated for buying instruments. Boucinha and Burlon (2020) indicated that negative spot-rates could happen in instances where the prime inflation rate is drastically low, as observed in the case of the Eurozone. The prime inflation rate in South Africa has been low over the estimation period; however, within the South African Reserve Bank target limit of 3-6 per cent. Some of the selected models gave negative spot-rates on some points of the curve, especially on shorter maturities; however, the frequency is very minimal. Negative spot-rates on the short-dated maturities are in line with market observations on some of the inflation-linked bonds where negative coupon-bearing yields were realised closer to their maturities [i.e. R211 and R189 bonds]. The results are consistent with the dynamics embedded on the short-dated maturities of the real spot-rate curve, which takes the characteristics of a money-market instrument. The results are in line with Reid (2009: 12) findings that 'in the short-term, the real interest rate is not constant (is strongly influenced by monetary policy)'.

Table 1: Number of times zero rates were realised

Maturity (yrs)	Vasicek	Linear parametric	Cubic splines	NS	DNS	RDNS	Svensson	DGNS	AFGNS
0.5	0	33	33	0	13	11	0	8	8
1	0	33	0	0	9	8	0	9	7
2	0	0	0	0	9	5	0	8	2
3	0	0	0	0	3	3	0	4	2
4	0	0	0	0	2	3	0	2	2
5	0	0	0	0	1	3	0	1	1
6	0	0	0	0	1	3	0	0	0
7	0	0	0	0	0	3	0	0	0
8	0	0	0	0	0	3	0	0	0
9	0	0	0	0	0	2	0	0	0
10	0	0	00	0	0	1	0	0	0
15	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0

Based on the output presented in Figure 3, most deviations are observed on shorter maturities of the spot-rate curve. The variations are consistent with the exercise done by Reid (2009), where real spot-rates in the long-end of the curve had a lower variation of around 2 per cent and 5 per cent. Lower variations are because spot-rates in the mid-to-long end of the curve are less volatile which is consistent with the finding made by Reid (2009, pp.14), that 'the interest rate at which these curves settled also did not fluctuate wildly from observation to observation'.

Figure 3: Abnormal rates realised

Figure 4: Perfect fit

Figure 5: Fitting Errors

Based on *Figure 4*, it could be observed that the arbitrage-free Vasicek (1977) one-factor model is giving deep discounts relative to all selected models. Discount factors of almost zero on the 30-year maturity point are considered significant for the South African instruments, thus failing this selection criterion. Based on the RMSE output on *Figure 5*, the Rotated Dynamic Nelson-Siegel (2018) model also gave a relatively higher frequency of a poor fit (i.e. higher Root Mean Squared Error (RMSE) estimates of more than 0.2 most of the time). Given that the original Dynamic Nelson-Siegel (2006) model gives a perfect fit over the entire estimation period, the effect of the rotation process did not improve the perfect fit. When compared to the Ghanaian study by Lartey and Li (2018), it could be concluded that the Svensson (1994) model has the capabilities of perfectly fitting the illiquid bond market with longer maturities.

Figure 6: Smoothness measure

It was indicated in Kikuchi and Shintani (2012, pp.23) that excessive unevenness in the spot-rate curve could be an indication of inappropriate interpolation which could have been driven by the estimation method with a higher degree of freedom. Based on the results on *Figure 6*, it was determined that the selected estimation methodologies gave an appropriate interpolation of the entire real spot-rate curve.

As part of the objectives of this study, forecasting the South African real spot-rate curve is also of interest. Given that economic forecasts are revised every quarter (National Treasury, 2018), to account for the most recent information; the forecast period is done for two periods (i.e. 3 and 6 months ahead). The forecasting process which is an estimation of the spot-rate at a given time in the future was used in this study. Based on the forecasting results on *Figure 7* and *Figure 8*, it could be noted that the AFGNS (2009) model predicts discounted zero-coupon bond prices in the short-end of the curve. Discounted bond prices at/closer to maturity are an indication of pricing arbitrage, and this contradicts the basics of bond pricing where bond prices converge to par pricing closer to bond maturity; as such, the bond prices are expected to trade closer to par in the short-end of the curve. On the spot-rates, it was observed that the volatility is expected on the short-end of the curve, which could be attributed to changes in monetary policy in line with Reid (2009, pp.12) findings. These volatilities in the short-end of the spot-rate curve are also observed in the study by Lartey and Li (2018, pp.10) on Ghanaian bond market.

Figure 7: One-step forecast (NB: Spot-rates are in percentage term)

Figure 8: Two-step forecast (NB: Spot-rates are in percentage term)
Source: Author's own work

To rule the estimation method as perfect, the model must also go through the backtesting process. Backtesting process entails the application of the selected models on a new dataset of the coupon-bearing bond prices to establish the capabilities of the model going forward.

Table 2: Forecasting Errors

RMSE	Time step 1	Time step 2
Linear	0,016	0,015
Cubic splines	0,017	0,014
NS	0,018	0,023
DNS	0,056	0,078
Svensson	0,018	0,029
DGNS	0,101	0,134
AFGNS	0,101	0,139

Based on Table 2, one can deduce that all the selected models are suitable for forecasting the South African real spot-rate curve; even though the models might lose their forecasting capabilities in forecasting for a more extended period. All selected models [except for the arbitrage-free generalized Nelson-Siegel (2009) and dynamic generalized Nelson-Siegel (2006)] gave a forecasting fit of at least 90 per cent (i.e. RMSE estimates of at-most 10 per cent) with linear parametric, cubic splines, Nelson-Siegel (1987) and Svensson (1994) models fitting at-least 98 per cent of the time. Even though the forecasted zero-coupon bond prices using all selected models are almost similar in the short-end of the curve, there is a significant difference on equivalent spot-rates in the short-end of the curve which is consistent with volatility dynamics embedded driven by monetary policy in the short-term. This volatility is also in line with Carvalho and Garcia (2019) finding where it was indicated that forecast values are not so accurate in the short-end of the curve which could be attributed to the instability of monetary policy and the volatility of short-term interest rates.

It was learned in this paper that two decades after the introduction of the first South African government inflation-linked bond, this market remained relatively illiquid. Due to inflation accrual on this bond instrument, investors prefer to hold the instrument to maturity with the aim of maximizing their returns. It was also learned that these liquidity issues have let to unavailability of data or real rates remaining constant over a period of time. Finding mathematical term-structure model to perfectly fit the South African real spot-rate curve remained critical to serve as a benchmark for the National Treasury's budgeting processes. These bond instruments contribute at-least twenty per cent of South African government funding in the domestic market (National Treasury, 2018, pp.83); as such, possible gaps in the pricing model might result with flawed budget figures. Different mathematical term-structure models were applied on the South African coupon-bearing inflation-linked bond prices to derive the real spot-rate curve. It was found that the rotated dynamic Nelson-Siegel (2018) model and the arbitrage-free Vasicek (1977) one-factor model were rejected based on negative/abnormal estimates and the poor fit. The linear parametric, cubic splines, Nelson-Siegel (1987), Dynamic Nelson-Siegel (2006), Svensson (1994), Dynamic Generalized Nelson-Siegel and Arbitrage-Free Generalized Nelson-Siegel (2009) models can capture at least 90 per cent of the movements in the spot-rate curve. The linear parametric model, cubic splines model, Nelson-Siegel (1987) model and Svensson (1994) model [in line with Reid (2009, pp.12)] are recommended as a perfect fit for forecasting the South African real spot-rate curve due to their in-sample backtesting performance of at least 98 per cent of the time.

Conclusions

The main objective of this study is to find a mathematical term-structure model for the South African real spot-rate curve given different models following the Nelson-Siegel (1987) framework. Given that the bond instrument contributes over 20 percent of the total government funding and the overall domestic debt, it is imperative to have a developed term-structure.

The South African inflation-linked coupon-bearing yield curve is categorised by several kinks due to only fewer points traded in the market and its illiquid nature. Given less significant regime switches in the South African monetary policy over the estimation period, it could then be noted that term-structure models with static estimation parameters gave a better fit. However, it could be noted that during higher market volatility, these assumptions could break and thus dynamic models [i.e. Dynamic Nelson-Siegel (2006) model] of which their performance was also significant, could be applied. However, for a solid conclusion to be derived, it is imperative to explore the performance of these models over a period of stressed market and economic conditions.

Budgeting deviations might often put the state on the wrong side of the rating agencies as debt service cost is recognised as one of the risk factors that are negatively affecting the credit rating of the state. It is

recommended that the results from this study be incorporated in the medium-term projections of South African budgeting process for more realistic pricing compared to the currently used econometrics term-structure models which have limitations in pricing the ultra-long-end of the curve.

References

- Bloomberg. (2018). Rates and bonds. <https://www.bloomberg.com/markets/rates-bonds>.
- Bold, S and Harris, L. (2018). Identifying monetary policy rules in South Africa with inflation expectations and unemployment. *World Institute for Development Economic Research*, wp-2018-43]. DOI:10.35188/UNU-WIDER/2018/485-8.
- Boucinha, M. and Burlon, L. (2020). Negative rates and the transmission of monetary policy. *ECB Economic Bulletin*, 3(2020). https://www.ecb.europa.eu/pub/economic-bulletin/articles/2020/html/ecb.ebart202003_02~4768be84e7.en.html.
- Carvalho, V and Garcia, M. (2019). A cross-sectional application of the Nelson-Siegel-Svensson model to several negative yield cases. *Cogent Economics and Finance*, 7(2019), 1. DOI: 10.1080/23322039.2019.1582319
- Christensen, J., Diebold, F. and Rudebusch, G. (2009). An arbitrage-free generalized Nelson-Siegel term structure model. *Econometrics Journal, Royal Economic Society*, 12(3), C33-C64. DOI: 10.3386/w14463.
- Diebold, F and Li, C. (2006). Forecasting the term structure of government bond yields. *Journal of Econometrics*, 130, 337-364. <https://www.sas.upenn.edu/~fdiebold/papers/paper49/Diebold-Li.pdf>.
- Garcia, J. and van Rixtel, A. (2007). Inflation-linked bonds form a central bank perspective. *European Central Bank*, Occasional Paper: 6. <https://www.ecb.europa.eu/pub/pdf/scpops/ecbocp62.pdf?2e9db05ab2507268ceef8d4b348420bd>.
- Gujarati, D. (2009). *Basic econometrics*. 5th ed. New Delhi: McGraw Hill.
- Ha, J., Kose, A. and Ohnsorge, F. (2019). Inflation in emerging and developing economies: evolution, drivers, and policies. *Washington, DC: World Bank*. DOI: 10.1596/978-1-4648-1375-7.
- Harman, S. and Mageza, N. (2009). South African government inflation-linked bond index: Index guide. *Barclays Bond Exchange of South Africa*, June(2009): 1-16. http://www.cranberger.com/sites/default/files/Cranberger_Index_And_Strategy/sa_inflation-linked_bond_index_guide_jun_09.pdf.
- Hull, J. (2009). *Options, futures and other derivatives*. 7th ed. University of Toronto: Pearson.
- Johannesburg Stock Exchange. (2012). The JSE zero-coupon yield curves: Methodology document, September 2012. <https://www.jse.co.za/content/JSEIndexCalculationMethodologyandConstructionItems/The%20JSE%20Zero%20Coupon%20Yield%20Curves.pdf>
- Kikuchi, K. and Shintani, K. (2012). Comparative analysis of zero coupon yield curve estimation methods using JGB price data. *Institute for Monetary and Economic Studies, Bank of Japan*, 12, E-04. <https://econpapers.repec.org/paper/imeimedps/12-e-04.htm>.
- Lartey, V. and Li, Y. (2018). Zero-coupon and forward yield curves for government of Ghana bonds. *SAGE Open*, July-September 2018, 1-15. DOI: 10.1177/2158244018800786.
- Naidoo, E, Nkuna, M. and Steenkamp, D. (2020). Developments in debt issuance costs of South African bank. *South African Reserve Bank*, Working Paper Series, WP/20/10. <https://www.resbank.co.za/Lists/News%20and%20Publications/Attachments/10157/WP%202010.pdf>.
- National Treasury. (2017). Holdings of domestic bonds: January 2017 and Outstanding Government Debt: 31 Oct 2016. <http://investor.treasury.gov.za/Pages/MarketIndicatorsandInformation.aspx>.
- National Treasury. (2018). Budget review 2018. <http://www.treasury.gov.za/documents/national%20budget/2018/default.aspx>.
- Nelson, C and Siegel, F. (1987). Parsimonious modeling of yield curves. *The Journal of Business*, 60(4), 473-89. DOI: 10.1086/296409.
- Nyholm, K. (2018). A rotated DNS model. *Economic Notes: Review of Banking, Finance and Monetary Economics*, 47(1), 113-123. DOI: 10.1111/econ.12094.
- Rafaelli, M. (2006). Pricing methodology for inflation linked bonds and inflation linked buy/sell-backs. *Bond Exchange of South Africa*, March 2006. <https://www.jse.co.za/content/JSE PricingItems/PricingMethodologyforInflationIndexedbonds.pdf>.

- Reid, M. (2009). Isolating a measure of inflation expectation for the South African financial market using forward interest rates. *South African Journal of Economics: Economic Society of South Africa*, 77(30), 399-413. <https://ideas.repec.org/p/sza/wpaper/wpapers81.html>.
- Suli, B. (2014). Yield curve modeling. MSc Actuarial and Financial Mathematics. Eotvos Lorand University. https://web.cs.elte.hu/blobs/diplomamunkak/msc_actfinmat/2015/suli_balazs_marton.pdf.
- Svensson, L. (1994). Estimating and interpreting forward interest rates: Sweden 1992 – 1994. *National Bureau of Economic Research*, Working Paper: 4871. DOI: 10.3386/w487.