IJFBS
Finance & Banking Studies

Finance & Banking Studies

IJFBS, VOL 10 NO 4 ISSN: 2147-4486
Contents available at www.ssbfnet.com/ojs
https://doi.org/10.20525/ijfbs.v10i4.1316

Value at Risk estimation using GAS models with heavy tailed distributions for cryptocurrencies*

Stephanie Danielle Subramoney
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa, ORCID ID: 0000-0003-3615-9585

Knowledge Chinhamu
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa, ORCID ID: 0000-0003-4296-7677

Retius Chifurira
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa, ORCID ID: 0000-0001-7889-3417

Abstract

Risk management and prediction of market losses of cryptocurrencies are of notable value to risk managers, portfolio managers, financial market researchers and academics. One of the most common measures of an asset's risk is Value-at-Risk (VaR). This paper evaluates and compares the performance of generalized autoregressive score (GAS) combined with heavy-tailed distributions, in estimating the VaR of two well-known cryptocurrencies' returns, namely Bitcoin returns and Ethereum returns. In this paper, we proposed a VaR model for Bitcoin and Ethereum returns, namely the GAS model combined with the generalized lambda distribution (GLD), referred to as the GAS-GLD model. The relative performance of the GAS-GLD models was compared to the models proposed by Troster et al. (2018), in other words, GAS models combined with asymmetric Laplace distribution (ALD), the asymmetric Student's t-distribution (AST) and the skew Student's t-distribution (SSTD). The Kupiec likelihood ratio test was used to assess the adequacy of the proposed models. The principal findings suggest that the GAS models with heavy-tailed innovation distributions are, in fact, appropriate for modelling cryptocurrency returns, with the GAS-GLD being the most adequate for the Bitcoin returns at various VaR levels, and both GAS-SSTD, GAS-ALD and GAS-GLD models being the most appropriate for the Ethereum returns at the VaR levels used in this study.

Keywords: *Bitcoin; Cryptocurrency; Ethereum; Generalized lambda distribution (GLD); GAS; Value-at-Risk*

JEL Classifications: *C2; C58; G32*

---

* This paper is derived from the MSc dissertation entitled "A comparison of GARCH-type models and GAS models for cryptocurrencies at University of KwaZulu-Natal".

Peer-reviewed Academic Journal published by SSBFNET with respect to copyright holders.

Introduction

There has been remarkable development in the cryptocurrency sphere over the past ten years. Bitcoin, one of the most widely traded cryptocurrencies, was first introduced and documented by Satoshi Nakamoto in 2008. Cryptocurrencies were established to promote the use of decentralized control so that electronic payments between individuals can be made without transacting via a third party (Ardia et al., 2018). This reduces transaction costs to an almost zero cost and/or price and allows for speedy buying and selling. Cryptocurrencies have always received a considerable amount of attention from academics and the finance industry since Bitcoin's inception. However, over the past few years the global interest in cryptocurrencies has grown exponentially, specifically from investors, central banks and governments. The enormous jump in value of cryptocurrencies such as Bitcoin and Ethereum from February 2017 to December 2017 has created an attraction to the analysis of this new market (Catania and Grassi, 2017). There is no central bank supervising the issuing of it and they have no exposure to common stock markets, hence factors like interest rates and inflation do not affect it. Volatility is highly prominent in cryptocurrencies and has been found to exhibit very similar behaviour to that of other financial time series such as foreign exchange returns (Liu et al., 2017).

Assessing the volatility of cryptocurrencies is vital, as they have found a place in financial markets as well as portfolio management (Dyhrberg, 2015). Bitcoin was initially perceived as a currency, as it was invented to be a medium of exchange; however, investors now use it as an investment tool. Many risks are affiliated with cryptocurrencies due to its unpredictable nature and because of the lack of adequate experience, regulators have not yet found the tools to regulate the market (Liu et al., 2017). It is essential to have an approximation of risk when trading financial assets. A basic tool that is employed to measure cryptocurrency risk is Value-at-Risk (VaR) which is defined as a measure of the maximum loss an investment can incur in a specific period at a specified level of confidence (Jorion, 2000). Since cryptocurrencies were only introduced in 2008, limited research and analysis of the statistical features of cryptocurrencies have been done. However, the studies that have been done since its inception allow one to have a better understanding of how these financial assets tend to behave.

Literature Review

Chan et al. (2017) evaluated various statistical distributions to model seven of the highest ranked cryptocurrencies adequately, but Ethereum was omitted. Daily data from 23 June 2014 to 28 February 2018 were used for the analysis. Value-at-risk (VaR) and expected shortfall measures of risk were used to evaluate the best fits. The distributions considered in the study were fitted to the returns of the cryptocurrencies. All the cryptocurrencies' returns were found to exhibit heavy tails, and the generalized hyperbolic distribution was found to outperform the other distributions, including the Laplace and Gaussian distributions. The volatility of Bitcoin was investigated by Sapuric and Kokkinaki (2014) by comparing it to the volatility of the exchange rates of major global currencies. The analysis was done using data from July 2010 to April 2014 and it was discovered that Bitcoin has high annualized volatility, but when transaction volume was taken into account, it proved to be more stable.

Ardia et al. (2018) used Markov-switching GARCH (MSGARCH) models to explore the existence of regime changes in the GARCH volatility dynamics of Bitcoin log returns. Daily Bitcoin mid-prices from August 2011 to March 2018 were used in the investigation. Since the MSGARCH models cater for structural breaks, it was shown that the Bitcoin returns do in fact exhibit regime changes in their volatility dynamics and by using the one-day-ahead VaR forecasts, the MSGARCH model was found to be the best fit when compared to the standard single-regime GARCH models. Katsiampa (2017) investigated the capabilities of GARCH models to capture the volatility of Bitcoin by using data from July 2010 to October 2016. Various GARCH-type models were tested, of which the autoregressive component GARCH (AR-CGARCH) model proved to be the model with the most optimal fit, based on goodness-of-fit measures. This result depicted how important it is to consider a short-run and long-run component of conditional variance.

Chu et al. (2017) modelled the daily data of seven of the most popular cryptocurrencies from June 2014 to May 2017 by using twelve GARCH models. Based on five different goodness-of-fit criteria which included the ability of the models to estimate VaR, the IGARCH and GJR-GARCH models obtained the best fits for modelling the volatility exhibited by the most popular cryptocurrencies. Bouoiyour et al. (2016) showed how volatility had decreased when daily Bitcoin data from the period 2010-2015 were compared with data from the beginning of 2015 by using an optimal GARCH model. This indicates that asymmetry in the Bitcoin market was prominent and that Bitcoin prices are thus more influenced by negative impacts than by positive ones. Liu et al. (2017) modelled Bitcoin data for a seven-year period, from 2010 to 2017, comparing the traditional Student's-t-distribution with a newly developed heavy-tailed distribution, the normal reciprocal inverse Gaussian under the GARCH framework. Based on the Akaike information criterion (AIC) and Bayesian information criterion, the Student's-t-distribution produced the best in-sample results.

Takaishi (2018) explored the statistical properties and multifractality of Bitcoin. 1-minute Bitcoin price index from January 2014 to January 2017 was utilized in the study and it was found that Bitcoin returns exhibit behaviour that is similar to that of the stylized facts of asset returns such as heavy-tailed return distribution, short-ranged serial correlations in returns and volatility clustering. It was also found that skewness tends to be negative at time scales that are shorter than one day and is zero at time scales that are longer than one week. The volatility asymmetry was explored by using GARCH, GJR and RGARCH models and no evidence of such was found. The Bitcoin series exhibited multifractality, which was investigated by using multifractal detrended fluctuation analysis.

Conrad et al. (2018) assessed the forecast performance of multiplicative volatility models by modelling the daily log returns on the S&P 500 for the 1971 to 2017 period. The forecasting abilities of the multiplicative GARCH models were compared for many forecast horizons and, based on the QLIKE loss, it was found that the multiplicative GARCH models incorporating financial and macro-economic variables improve on the heteroscedastic autoregressive (HAR) model. Equities of the NASDAQ-100 financial index series from January 2000 to December 2011 were modelled by Chalabi et al. (2012) by introducing new parameterizations of the GLD where the median and interquartile range were found to be interchangeable with the location and shape parameters of the distribution. This GLD proved to be advantageous in modelling financial returns as the GLD family has the ability to accommodate a great range of distribution shapes. The robustness of the GLD thus allows for a single distribution to be used to model the data from various asset classes. Corrado (2001) modelled non log-normal security price distributions by using the GLD. It was found that the flexibility of the GLD and its several other advantages make this distribution ideal to employ for numerous financial applications, for example approximation of option-implied state price densities and risk analyses based on Monte Carlo simulations.

Troster et al. (2018) modelled and forecasted the risks and returns of Bitcoin for the period July 2010 to April 2018 by using heavy-tailed GARCH models and the generalized autoregressive score (GAS) models with underlying distributions that include the normal, Student's-t and the asymmetric Student's-t (AST) with two decay parameters. The AIC and BIC criteria were used to assess the goodness-of-fit of the models and three backtesting procedures of 1%-VaR forecasts were performed to evaluate the VaR model specifications. The GAS models with heavy-tailed underlying distributions always outperformed the normally distributed GARCH and GAS models. The GAS-AST produced the best conditional and unconditional coverage for the 1% forecasts. Table 1 summarises some previous studies on modelling and analysing risk in cryptocurrencies. Thus, in this study we extend the work of Troster et al. (2018) by combining the GAS model with the GLD and by assessing the robustness of the proposed model at different VaR levels.

To the best of our knowledge, there are limited studies using GAS models with heavy tailed distributions, for example generalized lambda distributions (GLD), the asymmetric Laplace distributions (ALD), asymmetric Student's-t distribution (AST) with two-tail decay and the skew Student's-t distribution (SSTD) to model cryptocurrency data. The results acquired here are compared with those of Troster et al. (2018).

Peer-reviewed Academic Journal published by SSBFNET with respect to copyright holders.

Table 1: Summary of related research on modelling and analysing risk in cryptocurrencies

Authors	Data	Models	Robust models
Troster et al. (2018)	Bitcoin (Daily data for the period 19 July 2010-16 April 2018)	GARCH, APARCH GJRGARCH, TGARCH CGARCH, NGARCH HGARCH, EGARCH GAS-N, GAS-tS GAS-SSTD, GAS-AST GAS-AST1	GAS-AST
Chan et al. (2017)	Bitcoin Dash Litecoin MaidSafeCoin Monero Dogecoin Ripple (Daily data for the period 23 June 2014-28 February 2017)	Student's-t Laplace Skew t Generalized hyperbolic Normal inverse Gaussian Generalized-t Skew Student's-t Asymmetric Student's-t Generalized t	Generalized Hyperbolic
Katsiampa (2017)	Bitcoin (Daily data for the period 18 June 2010-1 October 2016)	GARCH EGARCH TGARCH APARCH CGARCH ACGARCH	AR-CGARCH
Chu et al. (2017)	Bitcoin Dash LiteCoin MaidSafeCoin Monero Dogecoin Ripple (Daily data for the period 23 June 2014-28 February 2017)	SGARCH, EGARCH, APARCH, IGARCH, TGARCH, AVGARCH, NAGARCH, ALLGARCH, GJR-GARCH, GARCH, NGARCH, CSGARCH	GJR-GARCH IGARCH
Liu et al. (2017)	Bitcoin (Daily data for the period 19 July 2010-23 July 2017)	GARCH GARCH-NRIG GARCH-SSTD	GARCH-NRIG
Takaishi (2017)	Bitcoin (1-min price index for the period 1 January 2014-31 December 2016)	GARCH GJR-GARCH RGARCH	GJR-GARCH
Ardia et al. (2018)	Bitcoin (Daily data for the period 18 August 2011-3 March 2018)	MSGARCH GJR-MSGARCH	GJR-MSGARCH

Peer-reviewed Academic Journal published by SSBFNET with respect to copyright holders.

Research and Methodology

This section introduces the theoretical frameworks and properties of the models used in the study.

GAS models

Generalized autoregressive score (GAS) models, proposed by Creal et al. (2013), are a set of observation-driven time series models. The GAS approach is distinguished from other observation-driven approaches in that the models' driving mechanism is the scaled score of the likelihood function. The GAS specification provides a framework for introducing time-varying parameters in a range of nonlinear models. It encompasses popular observation-driven models such as the GARCH models, autoregressive conditional density models (ACD) introduced by Engle and Russell (1998), autoregressive conditional multinomial models (ACM) of Rydberg and Shephard (2003), and Poisson models for count data.

We define $Y^t = \{y_1, y_2, \cdots, y_t\}$, $F^t = \{f_0, f_1, \cdots, f_t\}$ and $X^t = \{x_1, x_2, \cdots, x_t\}$ where X_t is a vector of independent variables.

Let Y_t be an $N \times 1$ random vector at time t with conditional distribution:

$$y_t \sim p(y_t | f_t, F_t; \theta), \quad (1)$$

where $F_t = \{Y^{t-1}, F^{t-1}, X^t\}$ for $t = 1,2, \cdots, n$ and θ is a vector of static parameters.

The time-varying parameter f_t is assumed to be updated by the following autoregressive equation:

$$f_{t+1} = \omega + \sum_{i=1}^{p} A_i s_{t-i+1} + \sum_{j=1}^{q} \beta_j f_{t-j+1}, \quad (2)$$

where ω is a vector of constants. A_i and B_j are matrices of coefficients that have suitable dimensions for i and j and are functions of θ. Furthermore, s_t is an appropriate function of past data, i.e. $s_t = s_t(y_t, f_t, F_t; \theta)$. The distinguishing characteristic of a GAS model is the choice of the innovation s_t as the local score, ∇_t. We specify the innovation as follows:

$$s_t = S_t \cdot \nabla_t \quad (3)$$

$$\nabla_t = \frac{\Delta \ln p(y_t | f_t, F_t; \theta)}{\Delta f_t} \quad (4)$$

$$S_t = S(t, f_t, F_t; \theta) \quad (5)$$

where $S(\cdot)$ is a matrix function.

These equations are defined to be the generalized autoregressive score function with orders p and q, i.e. GAS (p, q). Updating the time-varying parameter, f_t, with the use of the score function is known for improving the model's fit in terms of the likelihood at time t, with the current position of f_t being realized. The score, ∇_t, is dependent on the complete density of the observations y_t, which differentiates the GAS model framework from many other observation-driven models found in past literature. The structure is highly flexible in a sense that different choices for the scaling matrix S_t can produce many available observation-driven models introduced in literature, as S_t influences how the score is used to update f_t (Creal et al., 2013).

The MLE process is used to estimate the parameters of the GAS models fitted to both cryptocurrencies data. The methods implemented for the GAS models in the statistical software platform R are available within the GAS package.

Generalized lambda distribution

The generalized lambda distribution (GLD), initially introduced by Ramberg and Schmeiser (1974), is one of the most essential generalized classes of distributions. It is a four-parameter (location, scale, kurtosis, and skewness) modification of Tukey's lambda distribution (Tukey, 1960) that can be manipulated to produce common statistical distributions, such as Gaussian, lognormal, uniform and Weibull, as special cases.
In this study we use the FMKL (Freimer–Mudholkar–Kollia–Lin, 1988) parameterization defined as

Peer-reviewed Academic Journal published by SSBFNET with respect to copyright holders.

$$Q(u) = \lambda_1 + \frac{\frac{u^{\lambda_3 - 1}}{\lambda_3} - \frac{(1-u)^{\lambda_4 - 1}}{\lambda_4}}{\lambda_2}, \tag{6}$$

where λ_1 is the location parameter, λ_2 is the scale parameter, and, λ_3 and λ_4 are defined as the shape parameters, i.e., λ_3 is a skewness parameter and λ_4 is a kurtosis parameter. The methods applied for the FMKL GLD models in the statistical software platform R are available within the package GLDEX.

Asymmetric Student's t distribution

The class of generalized Student's t distribution that was proposed possesses one skewness parameter and two tail parameters as a skewness parameter principally controls the asymmetry of the central part of a distribution. This class of the Student's t distribution is known as the asymmetric Student's t- distribution (AST) and provides the potential to enhance the ability to fit and forecast empirical data in the tail regions which is essential to risk management and other financial econometric applications. Its probability density function is described by,

$$f(y;\alpha,\nu_1,\nu_2) = \begin{cases} \frac{\alpha}{\alpha^*} K(\nu_1)\left[1 + \frac{1}{\nu_1}\left(\frac{y}{2\alpha^*}\right)^2\right]^{-\frac{\nu_1+1}{2}} & , y \leq 0 \\ \frac{1-\alpha}{1-\alpha^*} K(\nu_2)\left[1 + \frac{1}{\nu_1}\left(\frac{y}{2(1-\alpha^*)}\right)^2\right]^{-\frac{\nu_2+1}{2}} & , y > 0, \end{cases} \tag{7}$$

where α is the skewness parameter and $\alpha \in (0,1)$. $\nu_1 > 0$ is the left-tail parameter and $\nu_2 > 0$ is the right-tail parameter. $K(\nu)$ and α^* are defined as,

$$K(\nu) \equiv \Gamma((\nu+1)/2)/\left[\sqrt{\pi\nu}\,\Gamma(\nu/2)\right], \tag{8}$$
$$\alpha^* = \alpha K(\nu_1)/[\alpha K(\nu_1) + (1-\alpha)K(\nu_2)]. \tag{9}$$

The density described in (7) is continuous and unimodal. Scale adjustments are made through α^* for the left and right parts of the density to guarantee continuity under changes of the shape parameters.

Asymmetric Laplace distribution

One of the most common symmetric distributions used for modelling data that exhibits leptokurtic behaviour is the classical Laplace distribution. However, this distribution is not appropriate for modelling data with an asymmetric empirical distribution and thus a class of distributions known as the asymmetric Laplace distributions was proposed. The asymmetric Laplace distribution (ALD), introduced by Kotz et al. (2001), portrays flexibility with regard to heavy tails and skewness found in a data set and thus makes it advantageous to model financial data. The ALD is given by,

$$f(y) = \frac{\kappa\sqrt{2}}{\tau(1+\kappa^2)} \begin{cases} \exp\left(-\frac{\kappa\sqrt{2}}{\tau}|y-\theta|\right) & , if \quad y \geq \theta \\ \exp\left(-\frac{\sqrt{2}}{\kappa\tau}|y-\theta|\right) & , if \quad y < \theta, \end{cases} \tag{10}$$

where θ is the location parameter, $\tau > 0$ is the scale parameter and $\kappa > 0$ is the skewness parameter.

Combining the GAS model with the GLD

The step-by-step method used to produce the GAS-GLD model is as follows:

- The first step is to fit a Gaussian GAS model to the Bitcoin and Ethereum returns data.
- The next step is to extract the standardized residuals from the fitted Gaussian GAS models. The generalized lambda distribution is then fitted to the extracted sets of standardized residuals.
- The last step is to use the Anderson-Darling test to assess the goodness-of-fit of the model.

Peer-reviewed Academic Journal published by SSBFNET with respect to copyright holders.

Value-at-Risk (VaR) and backtesting

Value-at-Risk (VaR) is the most commonly used risk metric in market risk management. It is a summary of the statistical measures of potential losses and is expressed as a confidence interval in units of a specific currency over a specific time period (Duffie and Pan, 1997). If $\hat{O}(\cdot)$ represents the cumulative distribution function (CDF) of the best fitting distribution then VaR can be defined as,

$$\text{VaR}\,(p) = \hat{O}^{-1}(p), \tag{11}$$

for $0 < p < 1$.

Generally, to backtest the adequacy of a model, the recursive approach of forecasting is employed (Marcellino, 2006). This method is also used to compare models in terms of VaR predictions. The aim of backtesting analysis is to assess the precision of the prediction by splitting the estimation and evaluation period. VaR backtesting procedures evaluate the true coverage of the unconditional and conditional left-tail of a log returns distribution (Ardia, Boudt and Catania, 2018). The correct unconditional coverage (UC) was first considered by Kupiec (1995) and the correct conditional coverage (CC) was first considered by Christoffersen (1998). In this study, we utilize the Kupiec likelihood ratio test for backtesting.

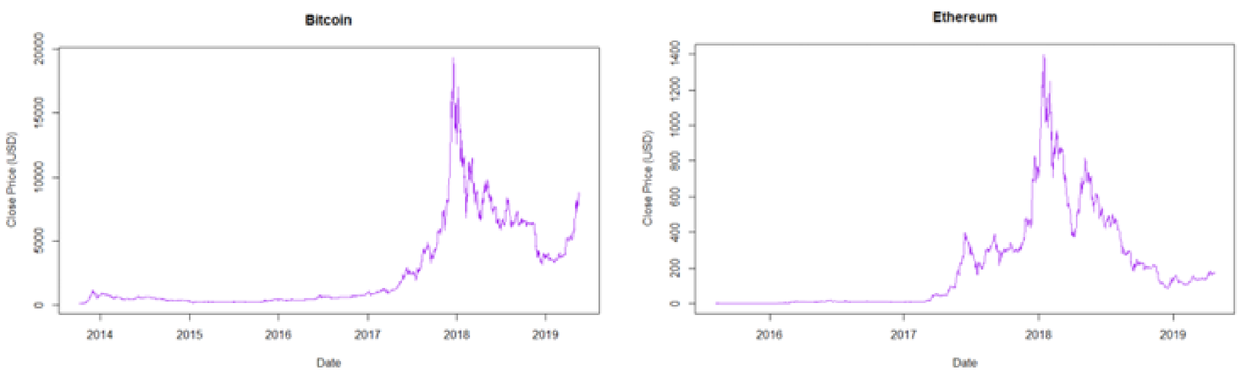
Empirical Results and Discussions

This section comprises the results that were produced by applying the models (described in previous section) to the Bitcoin and Ethereum datasets.

Data source and description

The data used in this analysis consist of the daily Bitcoin and Ethereum closing prices in United States dollars (USD). Different time periods were chosen for the two sets of cryptocurrencies due to data availability constraints. There are 2049 daily observations for the Bitcoin closing prices for the period 06/10/2013 to 28/05/2019 and 1355 daily observations for the Ethereum closing prices for the period 07/08/2015 to 22/04/2019.

The sets of data were obtained from: https://www.cryptodatadownload.com/data/northamerican/.In order to understand the properties of the data used, exploratory data analysis was utilized.



(a) (b)

Figure 1: Time series plots of (a) daily Bitcoin prices (USD) for the period 6 October 2013 to 28 May 2019 and (b) daily Ethereum prices (USD) for the period 7 August 2015 to 22 April 2019.

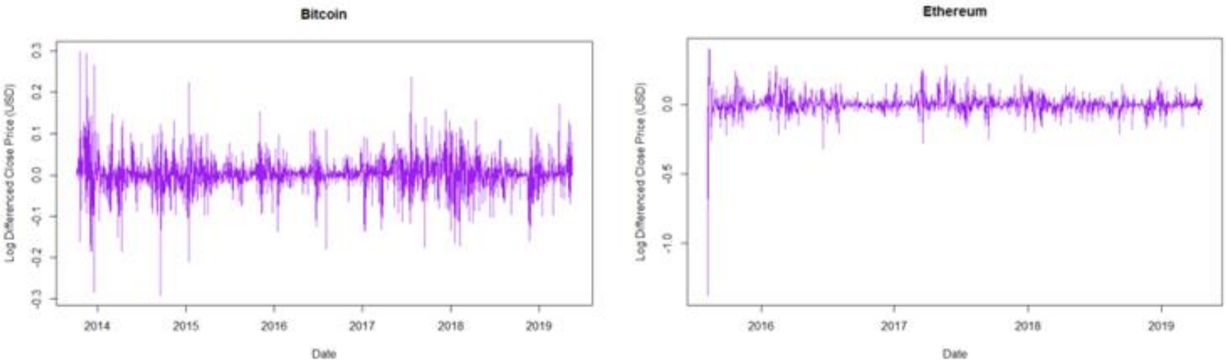
Figure 1 depicts the general trend that was observed by the daily Bitcoin and Ethereum prices for the specified period. It appeared that the prices fluctuated at relatively low amounts for Bitcoin from 2014 to early 2017 and for Ethereum from 2015 to early 2017. Thereafter, the close prices for both cryptocurrencies began to display a particular momentum and trend upwards until it peaked in 2018, after which exhibited an unsteady downward movement. Such behaviour suggests a series of non-stationarity due to the observed

Peer-reviewed Academic Journal published by SSBFNET with respect to copyright holders.

trends, which implies non-constant means and high variability. Investors are interested in returns on their investments and therefore, the closing prices data were then transformed to produce log-returns,

$$r_t = \ln \frac{p_t}{p_{t-1}}$$

where r_t is the log return, p_t is the closing price of the cryptocurrency at time t, and p_{t-1} is the closing price of the cryptocurrency at time $t-1$. Figure 2 shows the time series plots of the log returns for both cryptocurrencies.



(a) (b)

Figure 2: Time series plots of (a) daily Bitcoin log returns for the period 6 October 2013 to 28 May 2019 and (b) daily Ethereum log returns for the period 7 August 2015 to 22 April 2019.

The time series plots of the returns in Figure 2 display a fluctuating pattern around zero for both the Bitcoin and Ethereum data. This implies that the return series are now stationary in mean for the two cryptocurrencies. However, the observed trend of high and low periods suggests time-varying variance, which indicates the presence of volatility clustering. Leverage effects also appear to be prominent, as negative shocks to returns increase volatility with a larger impact than that of positive shocks.

Table 2: Descriptive statistics of the log returns of the daily Bitcoin and Ethereum prices

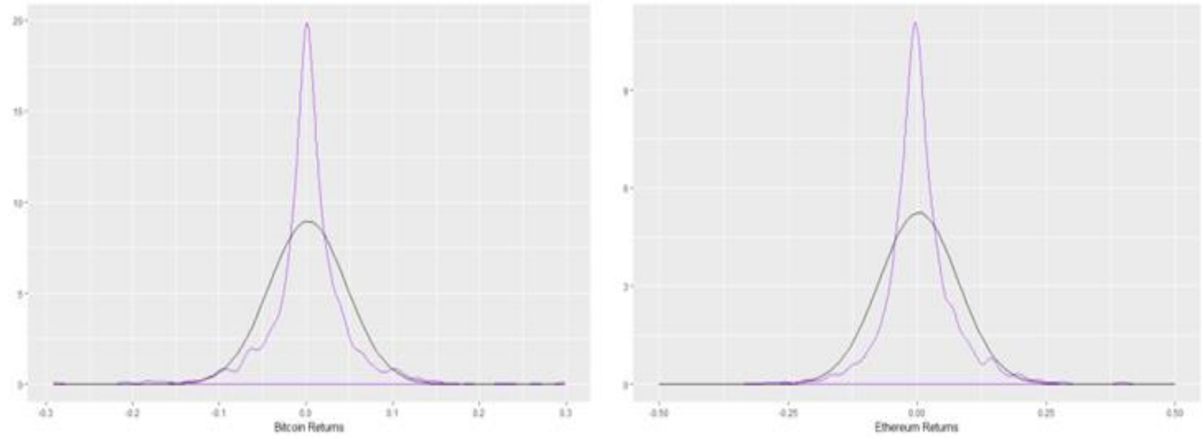
Statistics	Bitcoin	Ethereum
Minimum	-0.2915	-1.3740
Maximum	0.2988	0.4035
Mean	0.0021	0.0030
STDEV	0.0445	0.0760
Skewness	-0.0331	-4.0598
Kurtosis	7.0819	80.8327

Table 2 shows the descriptive statistics of the log returns of the daily Bitcoin and Ethereum prices. The mean of the Bitcoin and Ethereum returns was found to be positive. This indicates that both sets of returns were somewhat increasing. Both cryptocurrencies appear to be negatively skewed, which implies that the left tail is larger than that of the right and, as reported, Ethereum is more negatively skewed than Bitcoin. This suggests that the losses of the returns are greater than the profits for both cryptocurrencies, with Ethereum having larger losses. Bitcoin's, as well as Ethereum's, excess kurtosis is positive indicating that the daily returns are heavy-tailed.

Figures 3 and 4 show the density and Q-Q plots of the log returns, respectively. The density and Q-Q plots of the cryptocurrencies show significant deviations from the normal plot and normal line, respectively. It can

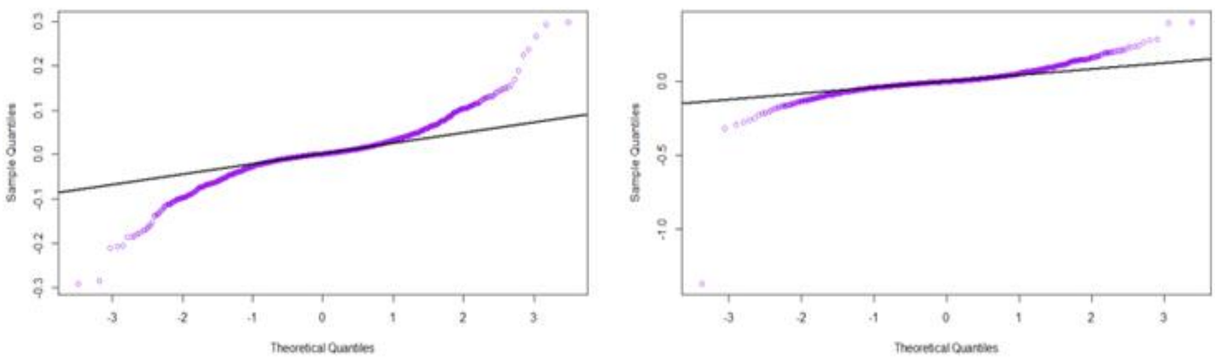
Peer-reviewed Academic Journal published by SSBFNET with respect to copyright holders.

be seen that the tails of both the Bitcoin and Ethereum log returns are heavier than those of the normal ones. The density plots are also observed to be substantially different from the normal distribution.



(a) (b)

Figure 3: Density plots of the daily (a) Bitcoin and (b) Ethereum log returns



(a) (b)

Figure 4: QQ plots of the daily (a) Bitcoin and (b) Ethereum log returns

Table 3: Formal tests of the log returns of daily Bitcoin prices (BTC/USD) and Ethereum prices (ETH/USD)

		Bitcoin		Ethereum	
Test		**Test Statistic**	***p*-value**	**Test Statistic**	***p*-value**
	ADF test				
	Case 1: No drift, no trend	-13.5000	0.0100	-12.4000	0.0100
	Case 2: Drift, no trend	-13.6000	0.0100	-12.5000	0.0100
	Case 3: Drift and trend	-13.6000	0.0100	-12.6000	0.0100
	PP test				
	Case 1: No drift, no trend	-2336.0000	0.0100	-1251.0000	0.0100
Stationarity	Case 2: Drift, no trend	-2326.0000	0.0100	-1246.0000	0.0100
	Case 3: Drift and trend	-2325.0000	0.0100	-1242.0000	0.0100
	KPSS test				
	Case 1: No drift, no trend	1.5100	0.0660	1.5800	0.0587
	Case 2: Drift, no trend	0.1300	0.1000	0.3000	0.1000
	Case 3: Drift and trend	0.1300	0.0720	0.1700	0.0340
Normality	Jarque-Bera test	4292.0000	< 0.0001	373480.0000	< 0.0001
	Shapiro-Wilk	0.8952	< 0.0001	0.7750	< 0.0001
Time variation	Cox-Stuart test	498.0000	0.3988	313.0000	0.0546
Leverage effects	Sign and size test	6.4384	0.0921	11.9897	0.0074
Autocorrelation	Ljung-Box test	7.7905	0.0506	3.2293	0.0723
ARCH effects	Box-Ljung test	7.6956	0.0055	9.9378	0.0191
	ARCH-LM test	169.1400	< 0.0001	580.2600	< 0.0001
Structural change	Chow test	3.069	0.0467	4.736	0.0089

The daily Bitcoin and Ethereum returns were found to be stationary as the resultant p-values of the ADF and PP tests of stationarity for all three cases, as given in Table 3, are less than 0.05, indicating that the null hypothesis of non-stationarity is rejected at a 5% level of significance. The KPSS test supports the two other tests of stationarity with test statistics that result in p-values greater than 0.05 and thus failing to reject the null hypothesis of stationarity. Tests for normality showed that the returns for both the cryptocurrencies are not normally distributed, as the p-values of the Jarque-Bera and Shapiro-Wilk test are less than 0.05, which suggests a rejection of the hypothesis of normality. This supports the deviations found in the plots in Figures 3 and 4. Time-variation was also investigated in both returns' data. As seen in Table 3, the p-values of the Cox-Stuart tests for Bitcoin and Ethereum are both greater than 0.05, implying that the null hypothesis of a non-monotonic trend is not rejected at a 5% level of significance. Thus, both sets of cryptocurrencies are independent and identically distributed. As mentioned above, the two cryptocurrencies' return series were

Peer-reviewed Academic Journal published by SSBFNET with respect to copyright holders.

found to be uncorrelated. However, strong ARCH effects are demonstrated by the data, as indicated by the Ljung-Box test performed on the squared returns and the ARCH-LM test.

A symmetric GARCH(1,1) model was fitted to the cryptocurrencies' data. The sign and size bias test was then applied and the joint effect test statistic p-values that were obtained were less than 0.10 for both Bitcoin and Ethereum. This implies that the null hypothesis of no asymmetric effects present should be rejected at a 10% level of significance. Hence, the GARCH(1,1) model failed to capture the asymmetry and thus there appears to be both a sign and size effect on future volatility of the returns. This implies that an asymmetric model will be required to capture the asymmetric effects exhibited. Bai-Perron breakpoint test (1998), it was found that there was a significant deviation in the Bitcoin return series around 23 August 2015. The data were then split into two parts, from which two regressions resulted. The Chow test for structural change was applied to this data and the resulting p-value was less than 0.05, indicating that the null hypothesis of no structural break is rejected. This implies that the series did, in fact, move away from the trend after the indicated time. Structural change for the Ethereum returns data around 29 January 2018 was detected. The exact approach applied to the Bitcoin returns was then utilized for this data. The p-value = 0.0089 suggests that the null hypothesis is to be rejected implying that the structural break identified is significant. The results of the exploratory data analysis are now summarized. From the exploratory data analysis, it can be concluded that the daily Bitcoin and Ethereum returns exhibited the following empirical properties: volatility clustering and non-linear dependence, heavy-tails, significant serial correlations in the absolute and squared returns, time variation, leverage effects, ARCH effects and structural change.

Fitting GAS models with heavy-tailed distributions

The first step to perform a GAS analysis is to specify an appropriate model. The conditional distributions that were of focus here for both sets of cryptocurrency data, were the skewed Student's-*t*, the asymmetric Laplace and the asymmetric Student's-*t* (with two-tailed decay parameter) distributions. A GAS model with a GLD conditional distribution was also considered. All specifications utilized one of the three scaling mechanisms to acquire adequate fits (Ardia, Boudt and Catania, 2019). A GAS (1, 1) model with the above listed distributions exhibited a time variation in one or more of the parameters for location, scale, skewness and shape (1 and/or 2) of the distribution. All parameters of the three models fitted to both sets of data were found to be statistically significant at a 5% level of significance. Thus, all models appear to have been adequate fits.

Table 4: Estimation of results of the Gaussian GAS (1,1) model for the Bitcoin and Ethereum daily returns

	Parameters	**Estimate**	**Std**	**p-value**
Bitcoin	$\widehat{\omega}_{\mu}$	0.0014	0.0008	0.0323
	$\widehat{\omega}_{\phi}$	-0.3310	0.0202	< 0.0001
	$\hat{a}_{\phi}$	0.1898	0.0166	< 0.0001
	$\hat{b}_{\phi}$	0.9469	0.0032	< 0.0001
Ethereum	$\widehat{\omega}_{\mu}$	6.046×10^{-5}	4.151×10^{-5}	0.0727
	$\widehat{\omega}\phi$	-1.031×10^{-1}	7.610×10^{-4}	< 0.0001
	$\hat{a}_{\mu}$	1.000×10^{-4}	1.053×10^{-13}	< 0.0001
	$\hat{a}_{\phi}$	1.000×10^{-4}	1.053×10^{-4}	< 0.0001
	$\hat{b}_{\mu}$	9.800×10^{-1}	4.854×10^{-11}	< 0.0001

A Gaussian GAS (1,1) model was fitted to both sets of data and the estimates are found in Table 4. The model appears to have been a good fit at a 10% level of significance. The residuals of these modelsWere then extracted and analysed.

Table 5: Descriptive statistics of the residuals extracted from the Gaussian GAS model fitted to the log returns of the daily Bitcoin (BTC/USD) and Ethereum (ETH/USD) prices

Statistics	Bitcoin	Ethereum
Minimum	-5.8473	-18.1154
Maximum	5.9044	5.2697
Mean	0.0080	0.00002
STDEV	0.9510	1.0000
Skewness	-0.2670	-4.0600
Kurtosis	4.8550	80.8443

The descriptive statistics of the residuals were obtained and is reported in Table 5. It appears that the mean of the residuals of both the cryptocurrencies are not significantly different from zero. The excess kurtosis of 4.8550 and 80.8443 for the Bitcoin and Ethereum residuals, respectively, suggest that heavy tails are, in fact, prominent. The results of the Jarque-Bera (p-value <0.0001) and Shapiro-Wilk tests (p-value <0.0001) of normality imply that the residuals' tails are heavier than that of the normal distribution. Fitting a GLD to the extracted residuals was then justified as the GLD accommodated for the leptokurtic behaviour exhibited. A GLD model was fitted to the Gaussian GAS residuals with FMKL parameterizations by using the MLE method. The estimates of the model are found in Table 6. The Anderson-Darling test results, as recorded in Table 6 implies that the GLD models were good fits at a 5% level of significance.

Table 6: GLD models fitted to the extracted Bitcoin and Ethereum residuals from the Gaussian GAS model.

Estimates	Bitcoin	Ethereum
$\hat{\lambda}_1$	0.0192	-0.0547
$\hat{\lambda}_2$	3.3740	4.0269
$\hat{\lambda}_3$	-0.3109	-0.2969
$\hat{\lambda}_4$	-0.2883	-0.4175

Table 7: Anderson-Darling tests of the GLD fitted to the residuals extracted from the Gaussian GAS model

	Test Statistc	*p*-value
Bitcoin	1.4154	0.1980
Ethereum	1.2821	0.2383

Estimating risk measures is essential in finance as this information is required by traders to examine the uncertainty associated with the future values of their portfolios and to take into account any potential losses. VaR is one of the standard risk measures and it was estimated at long and short positions. Traders at a short position (traders who are selling) will generally incur a loss when price increases, whereas and traders at a long position (traders who are buying) will incur a loss when the price drops. Table 8 reports the VaR estimates for both short and long positions, which are affiliated with the right and left quantiles of the returns' distribution, respectively. The 1%, 2.5%, 5%, 95%, 97.5% and 99% risk levels were taken into account.

Table 8: VaR estimates for the returns at long and short positions

	Distribution	Long position			Short position		
		1%	**2.5%**	**5%**	**95%**	**97.5%**	**99%**
Bitcoin	GAS-SSTD	-0.1089	-0.0862	-0.0691	0.0736	0.0907	0.1134
	GAS-AST	-0.0417	-0.0342	-0.0278	**0.0358**	**0.0421**	**0.0496**
	GAS-ALD	-0.1215	-0.0927	-0.0709	0.0740	0.0958	0.1246
	GAS-GLD	**-3.0153**	**-2.0215**	**-1.4318**	1.4141	1.9612	2.8660
Ethereum	GAS-SSTD	-0.1858	-0.1463	-0.1169	0.1203	0.1496	0.1891
	GAS-AST	-0.1442	-0.1096	-0.0861	**0.1100**	**0.1325**	**0.1594**
	GAS-ALD	-0.2107	-0.1615	-0.1242	0.1231	0.1603	0.2095
	GAS-GLD	-2.4987	-1.7129	-1.2411	1.4153	2.1191	3.4162

[a] Bold values are the lowest VaR estimates.

Peer-reviewed Academic Journal published by SSBFNET with respect to copyright holders.

It was found that for the Bitcoin returns, the GAS-GLD produced the smallest VaR estimates for the long positions and the highest VaR estimates for the short positions. The GAS-AST distribution has the largest VaR estimates for the long positions and the smallest VaR estimates for the short positions. For the Ethereum returns, similarly to the Bitcoin returns, the GAS-AST distribution has the highest VaR estimates for the long positions and the smallest VaR estimates for the short positions. The GAS-GLD produced the largest VaR estimates for the other two short positions of 97.5% and 99%, respectively.

In-sample backtesting

In-sample backtesting was performed by using the Kupiec likelihood ratio test (1995) to assess the model adequacy in estimating the VaR estimates. The *p*-values of the Kupiec likelihood ratio test at the different risk levels for the in-sample data are found in Table 10. The most robust models were chosen, based on the highest *p*-values at a specific level. The p-values less than 0.05 indicate that the null hypothesis of model adequacy was rejected at the levels considered. For Bitcoin, it is observed that the GAS-AST model rejected the null hypothesis at all specified levels. The GAS-ALD model had the highest *p*-value at the 1% long position. The GAS-GLD outperformed the rest of the models at the 2.5% level. The GAS-SSTD performed the best at the 5% level and shared the highest *p*-value with the GAS-ALD model at the short position of 95% level. The GAS-GLD appears to have had the best performance at the 97.5% level and the GAS-ALD at the 99% level. For the Ethereum returns, the GAS-SSTD and GAS-GLD had the highest *p*-value at the 1% level of the long position. The GAS-GLD model was the most robust model at the other two levels of the long position. The GAS-SSTD performed the best at the 95% level. The GAS-ALD had the highest *p*-value at the 97.5% and 99% levels.

Table 9: In-sample VaR backtesting for the Bitcoin and Ethereum returns

	Distribution	**Long position**			**Short position**		
		1%	**2.5%**	**5%**	**95%**	**97.5%**	**99%**
Bitcoin	GAS-SSTD	0.0181	0.0448	**0.5122**	**0.7289**	0.0326	0.0181
	GAS-AST	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	GAS-ALD	**0.5831**	0.5950	0.3881	**0.7289**	0.2809	**0.5831**
	GAS-GLD	0.4259	**0.6944**	0.2901	0.0664	**0.4541**	0.2013
Ethereum	GAS-SSTD	**0.6680**	0.1065	0.0042	**0.8315**	0.5890	0.0994
	GAS-AST	0.0025	0.0054	0.1355	0.1693	< 0.0001	< 0.0001
	GAS-ALD	0.1867	0.0049	0.0001	0.7347	**0.7452**	**0.6680**
	GAS-GLD	**0.6680**	**0.9792**	**0.5138**	0.3097	0.3870	0.0022

[b] Bold values are largest *p*-values.

Out-of-sample backtesting

Out-of-sample backtesting was applied by using the unconditional coverage (UC) test of Kupiec (1995) and the results are found in Table 10. The aim of out-of-sample backtesting analysis is to assess the precision of the predictive ability of the VaR model by splitting the estimation and evaluation period. The out-of-sample period for the Bitcoin returns is from 28 May 2019 to 3 February 2020 and for the Ethereum returns from 22 April 2019 to 3 February 2020. In Table 10, the Kupiec p-values are presented for both positions. The highlighted values in Table 12 indicate that the null hypothesis of correct model specification should be rejected at the specified a-quantile level. It is observed that the null hypothesis was rejected for the GAS-AST model fitted to the Bitcoin returns at all a-quantile levels. Thus, it seems that the GAS-AST model is not adequate and has low predictability power at these levels which is contrasting to the results of Troster et al. (2018), who found that the GAS-AST model produced the best out-of-sample forecasts. For the Bitcoin returns, the GAS-SSTD, GAS-ALD and GAS-GLD appear to have had adequate VaR models at all levels of the long and short positions considered, as none of the VaR models rejected the null hypothesis of model adequacy at these levels. At the 1% level, the GAS-SSTD had the highest p-value. The GAS-GLD outperformed the rest at the 2.5% and 97.5% levels. The GAS-ALD appears to have had the best performance at the 5% level of the long position and the 95% and 99% levels of the short position. For the Ethereum returns, at the 1% and 2.5% long positions, all the VaR models appear to be adequate as the models failed to reject the null hypothesis. The GAS-SSTD was the most robust models at these levels, as well as the 5% long position. It is observed that the GAS-AST and the GAS-ALD had similar p-value outputs for the long position and the 95% level of the short position. The GAS-GLD outperformed the rest of the

Peer-reviewed Academic Journal published by SSBFNET with respect to copyright holders.

models considered at the 95% level. The GAS-SSTD outperformed the rest of the models at the 97.5% and 99% levels.

Table 10: Out-of-sample VaR backtesting for the Bitcoin and Ethereum returns

		Long position			Short position		
	Distribution	**1%**	**2.5%**	**5%**	**95%**	**97.5%**	**99%**
Bitcoin	GAS-SSTD	**0.7680**	0.5866	0.2738	0.6367	0.7813	0.7327
	GAS-AST	< 0.0001	< 0.0001	< 0.0001	< 0.0001	< 0.0001	< 0.0001
	GAS-ALD	0.2732	0.5866	**0.9084**	**0.9084**	0.5866	**0.7680**
	GAS-GLD	0.2732	**0.7813**	0.6907	0.3446	**0.9030**	0.2732
Ethereum	GAS-SSTD	**0.9247**	**0.9696**	**0.3651**	0.0304	**0.1998**	**0.5968**
	GAS-AST	0.5968	0.3997	0.0304	0.0011	0.0038	0.0169
	GAS-ALD	0.5968	0.3997	0.0304	0.0011	0.0787	0.2053
	GAS-GLD	0.8197	0.8755	0.6027	**0.1389**	0.0641	0.0005

[b] Bold values are the largest *p*-values.

Conclusion

Considering the growing interest in cryptocurrencies, it is of great significance to identify models that have the ability to forecast the risks associated with investment opportunities correctly. This study analysed and modelled the statistical characteristics exhibited by Bitcoin and Ethereum returns. GAS models with heavy-tailed distributions were implemented to accommodate the features of these cryptocurrencies. The results of the show that the GAS model with heavy-tailed innovation distributions are, in fact, appropriate for modelling cryptocurrency returns, as the models tested here have the ability to encapsulate the properties that define them, with the GAS-ALD and GAS-GLD being the best for the Bitcoin returns, and the GAS-SSTD, GAS-GLD and GAS-ALD models the best for the Ethereum returns. The GAS and GARCH models were also found to have adequate predictability power, with all models performing well except the GAS-AST model. With regard to the GAS-AST model, the findings of this study is in contrast to Troster et al., (2018) who found that this model presented the best out-of-sample forecast performance for the Bitcoin returns. Following April 2018, it was observed that there was fluctuating decline in the close prices and the difference in the findings could thus be attributed to the fact that different time periods were used. Due to the increasing popularity, innovative characteristics, simplicity and transparency of cryptocurrencies, this information could be of interest to risk managers, investors, as well as academics (Katsiampa, 2017).

References

Ardia, D., Bluteau, K., & Ru ̈ede, M. (2019). Regime changes in Bitcoin GARCH volatility dynamics. Finance Research Letters, 29(C), 266-271. https://doi.org/10.1016/j.frl.2018.08.009

Ardia, D., Boudt, K., & Catania, L. (2018). Downside risk evaluation with the R Package GAS. The R Journal, 10(2), 410-421. http://dx.doi.org/10.32614/RJ-2018-064

Bouoiyour, J., Selmi, R., Tiwari, A., & Olayeni, O. (2016). What drives Bitcoin price? Economics Bulletin, 36, 843-850.

Catania, L., & Grassi, S. (2017). Modelling Cryptocurrencies Financial Time Series. SSRN Electronic Journal.

Chalabi Y., Scott, D.J., & Wuertz, D. (2012). Flexible distribution modelling with the generalized lambda distribution. MPRA Paper No. 43333.

Chan, S., Chu, J., Nadarajah, S., & Osterrieder, J. (2017). A Statistical Analysis of Cryptocurrencies. Journal of Risk and Financial Management 10(2), 12. https://doi.org/10.3390/jrfm10020012

Christoffersen, P. (1998). Evaluating interval forecasts. International Economic Review, 39(4): 841-862.

Chu, J., Chan, S., Nadarajah, S., & Osterrieder, J. (2017). Volatility estimation for Bitcoin: A comparison of GARCH models. Journal of Risk and Financial Management 11, 23. https://doi.org/10.3390/jrfm10040017

Conrad, C., Custovic, A., & Ghysels, E. (2018). Volatility estimation for Bitcoin: A comparison of GARCH models. Journal of Risk and Financial Management, 11, 23.

Corrado, C.J. (2001). Option Pricing Based on the Generalized Lambda Distribution. Journal of Futures

Market. http://dx.doi.org/10.2139/ssrn.248696

Creal, D., Koopman, S.J., & Lucas, A. (2013). Generalized autoregressive score models with applications. Journal of Applied Econometrics, 28, 777-795. https://doi.org/10.1002/jae.1279

Duffie, D., & Pan, J. (1997). An Overview of Value at Risk. Testing for unit roots: An empirical investigation, 4(3), 7-49. https://doi.org/10.3905/jod.1997.407971

Dyhrberg, A. (2015). Bitcoin, gold and the dollar – A GARCH volatility analysis. Finance Research Letters, 16, 223-236. https://doi.org/10.1016/j.frl.2015.10.008

Engle, R. F., & Russell, J.R. (1998). Autoregressive Conditional Duration: A New Model for Irregularly Spaced Transaction Data. Econometrica 66(5), 1127-1162. https://doi.org/10.2307/2999632

Jorion, P., (2000). Value at Risk: The New Benchmark for Managing Financial Risk.

Katsiampa, P. (2017). Volatility estimation for Bitcoin: A comparison of GARCH models. Economics Letters 158(C), 3-6. https://doi.org/10.1016/j.econlet.2017.06.023

Kotz, S., Kozubowski, T.J. & Podgórski, K. (2001) Asymmetric Multivariate Laplace Distribution. In: The Laplace Distribution and Generalizations. Birkhäuser, Boston, MA. https://doi.org/10.1007/978-1-4612-0173-1_7

Kupiec, P. (1995). Techniques for verifying the accuracy of risk measurement models. Journal of Derivatives, 2, 173-184. https://doi.org/10.3905/jod.1995.407942

Liu, R., Shao, Z., Wei, G., & Wang, W. (2017). Volatility estimation for Bitcoin: A comparison of GARCH models. Journal of Accounting, Business and Finance Research, 1, 71-75. https://doi.org/10.20448/2002.11.71.75.

Marcellino, M. (2006). Leading Indicators. Handbook of Economic Forecasting, 1, 879-960. https://doi.org/10.1016/S1574-0706(05)01016-5

Ramberg, J. S., & Schmeiser, B. W., (1974). An approximate method for generating asymmetric random variables. Commun. ACM 17(2), 78–82. https://doi.org/10.1145/360827.360840

Sapuric, S. & Kokkinaki, A. (2014). Bitcoin Is Volatile! Isn't that Right? Lecture Notes in Business Information Processing 183:255-265. Conference: International Conference on Business Information Systems. http://dx.doi.org/10.1007/978-3-319-11460-6_22

Takaishi, T. (2018). Statistical properties and multifractality of Bitcoin. Physica A: Statistical Mechanics and its Applications 506. https://doi.org/10.1016/j.physa.2018.04.046

Troster, V., Tiwari, A.K., Shahbaz, M., & Macedo, D.N. (2018). Bitcoin returns and risk: A general GARCH and GAS analysis. Finance Research Letters. https://doi.org/10.1016/j.frl.2018.09.01

Peer-reviewed Academic Journal published by SSBFNET with respect to copyright holders.